ADVANCES IN COMPUTATIONAL MULTIBODY DYNAMICS
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APPLICATION OF QUATERNION ALGEBRA TO THE EFFICIENT COMPUTATION OF JACOBIANS FOR HOLONOMIC-RHEONOMIC CONSTRAINTS

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Abstract. *This work deals with an approach to the multibody simulation based on the Lagrangian "augmented" formulation, where the adoption of quaternions as rotational coordinates can lead to interesting results. The properties of quaternion algebra let us obtain a single analytical general-purpose formulation of constraint's jacobians which can be used for a wide class of holonomic joints, like spherical joints , revolute joints, cylindrical joints, 'point on line' and many others, up to 64.*
Despite the apparent complexity of this analytical approach, the formulas has been arranged in an optimal way which allows easy run-time simplifications and fast, efficient calculus. Moreover, given that most constraints can be represented with a single formulation, the method fits well into an object-oriented approach. We implemented a multibody software in C++ language on the basis of these theoretical results.

1 INTRODUCTION

The most common classification among multibody system formulations relies on the type of coordinates adopted in the equations. Using a maximal set of coordinates, as in our method, all the translational and rotational coordinates of rigid bodies are taken into account in the differential dynamical equations, while all the constraints between rigid bodies are translated into additional algebraic equations (the so-called *Lagrangian "augmented" formulation*).

Either in case of DAE or ODE numerical solutions [1,4], the lagrangian formulation requires the computation of the constraint's jacobians, as well as other complex terms resulting from the differentiation of constraint equations with respect to time and generalized coordinates.

Performing a numerical differentiation to obtain jacobians and the above mentioned terms, a computational overhead may take place, especially in the circumstance of complex spatial mechanisms with many couplings.

On the other hand, an analytical formulation of jacobians could be accomplished off-line (either with automatic symbolic differentiation or by hand) in order to improve speed and precision, but this method would lose generality, in the sense that each type of constraint would need its own analytical differentiation. Therefore it would be interesting to develop a method to get the analytical derivation of constraints, which is either fast and general in its application, comprehending a wide class of holonomic spatial constraints into a compact formulation.

Thus, we created a general-purpose constraint equation ("lock" equation) which imposes a condition of mutual position and rotation between two reference frames belonging to rigid bodies, where both position and rotation can be expressed in rheonomic (time-dependant) terms. Consequently we obtained a wide class of spatial couplings and actuators, simply by suppressing some of the six constraints of this "lock" formulation, and by providing adequate motion laws when needed.

The choice of quaternions as rotational coordinates rigid bodies let us work out the analytical derivation of such "lock" equation in a coherent and compact form, thank to the handiness of the quaternion algebra.

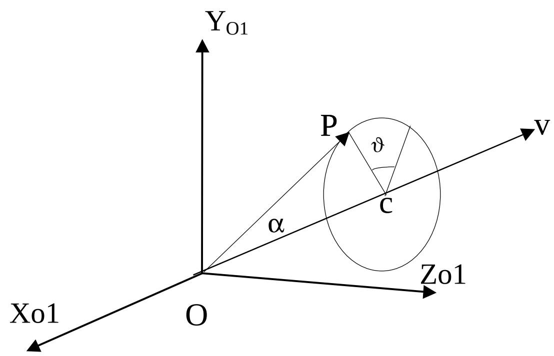
2 OVERVIEW OF ROTATIONAL COORDINATES

For each body in the system, some kind of coordinates are needed to represent the rotation in three-dimensional space. Usually this task is accomplished via three angles (Eulero's angles, Cardano's angles, HPB angles, etc.) which indicate the rotation of body's frame about absolute reference, through specific sequences of rotations about the three body's axis. Hence, the 3x3 matrix of rotation of a frame is a function of three angles $\mathbf{a}$={r,s,t}:

$$[\mathbf{\Lambda}] = [\mathbf{\Lambda}(\mathrm{r},\mathrm{s},\mathrm{t})] \tag{1}$$

Among the most relevant problems concerned with whatever sets of three parameters/angles[7], there's the fact that all the corresponding inverse transformation $\mathbf{a}$= {a,b,c} = $f([\mathbf{\Lambda}])$ may exhibit some singularities. This means that there may be some alignments of bodies where one of the three angles can't be obtained, and passing near these configurations may cause numerical difficulties as soon as such angular coordinates are used in the formulation of equations.

Another way to represent the rotation in space is the set of four Eulero's parameters, which is not subject to the problem of singularities. The four parameters $\mathbf{q}=\{\vartheta_1 \ \vartheta_2 \ \vartheta_3 \ \vartheta_4\}$ are expressed as a function of the rotation axis $\mathbf{v}$ and angle of rotation θ about $\mathbf{v}$, as shown in figure.



The formulation of the four parameters is the following:

$$\vartheta_0 = \cos\left(\frac{\theta}{2}\right) \qquad \vartheta_1 = \mathrm{v}_x \sin\left(\frac{\theta}{2}\right) \tag{2 a,b}$$

$$\vartheta_2 = \mathrm{v}_y \sin\left(\frac{\theta}{2}\right) \qquad \vartheta_3 = \mathrm{v}_z \sin\left(\frac{\theta}{2}\right) \tag{2 c,d}$$

The matrix [**Λ**] can be obtained as a function of the above parameters[1]:

$$[\mathbf{\Lambda}] = \begin{bmatrix} 2\left[(\vartheta_0)^2 + (\vartheta_1)^2\right] - 1 & 2(\vartheta_1\vartheta_2 - \vartheta_0\vartheta_3) & 2(\vartheta_1\vartheta_3 + \vartheta_0\vartheta_2) \\ 2(\vartheta_1\vartheta_2 + 2\vartheta_0\vartheta_3) & 2\left[(\vartheta_0)^2 + (\vartheta_2)^2\right] - 1 & 2(\vartheta_2\vartheta_3 - \vartheta_0\vartheta_1) \\ 2(\vartheta_1\vartheta_3 - \vartheta_0\vartheta_2) & 2(\vartheta_2\vartheta_3 + \vartheta_0\vartheta_1) & 2\left[(\vartheta_0)^2 + (\vartheta_3)^2\right] - 1 \end{bmatrix} \tag{3}$$

and the inverse transformation **q**=*f*([**Λ**]), which is not prone to singularities, can be obtained as well[5].

3 QUATERNIONS

Sir William Hamilton developed quaternion algebra in 1843, after long researches on hypercomplex numbers [9]. Since then, quaternions have been widely used in mechanics, because they can easily represent rotations of reference frames in space [2,5] as soon as a correspondence between them and the four Eulero's parameters is built.

Before developing our multibody equations, we must introduce some basic quaternion algebra.

Quaternions are four-dimensional hypercomplex numbers, with one real part an three imaginary parts:

$$\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \tag{4}$$

$$\mathbf{q} \in \left\{\Re^1, \Im^3\right\}$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, **ij = k**, **ji = -k**, with cyclic permutation $\mathbf{i} \to \mathbf{j} \to \mathbf{k} \to \mathbf{i}$.

A quaternion can be written either in its four-dimensional vectorial form

$$\mathbf{q} = \{q_0 \quad q_1 \quad q_2 \quad q_3\}^T \tag{5}$$

or in its *scalar / imaginary-vectorial* notation [s,**v**], that is:

$$\mathbf{q} = [s, \mathbf{v}] \; = \; s + v_x\,\mathbf{i} + x_y\,\mathbf{j} + v_z\,\mathbf{k} \tag{6}$$

Using the above mentioned notation, some interesting rules can be enunciated.
The *conjugate* **q'** of a quaternion comes from the quaternion **q** where the sign of the vectorial-imaginary part has been changed:

$$\mathbf{q} = [s, \mathbf{v}] \qquad \mathbf{q}' = [s, -\mathbf{v}] \tag{7}$$

The *euclidean norm* of the quaternion **q** is defined as follows:

$$\|\mathbf{q}\| = \left(q_0{}^2 + q_1{}^2 + q_2{}^2 + q_3{}^2\right)^{1/2} \tag{9}$$

and a quaternion whose norm equals one is called *unit quaternion*.
The *product* between two quaternions is given by the following formula:

$$\mathbf{q}_1\mathbf{q}_2 = \left(s_1 s_2 - \mathbf{v}_1\mathbf{v}_2 \quad , \quad s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2\right) \tag{10}$$

Note that quaternion product is non-commutative, as it can be seen from the vectorial part of the formula, where a cross-product between the two imaginary-vectorial parts is performed.
A so-called *pure quaternion* has only the imaginary part:

$$\mathbf{q} = [0, \mathbf{v}]$$

Now we can write the four dimensional vector of Eulero's parameters as a quaternion:

$$\vartheta = \begin{Bmatrix} \vartheta_0 \\ \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{Bmatrix} \quad \Rightarrow \qquad \mathbf{q} = (s, \mathbf{v}) \; = (\cos(\vartheta/2) , \mathbf{n} \sin(\vartheta/2)) \tag{11}$$

where ϑ represents the angle of rotation about the axis **n .**

From formulae (2 a,b,c,d), with some trigonometric calculus, one can find that only quaternions with unitary euclidean norm can be valid sets of Eulero's parameters, that is, if **q** is a quaternion which represent a rotation of a frame in space, via four Eulero parameters, then

$$\|\mathbf{q}\| = 1 \tag{12}$$

This means that Elero's parameters lies on the hyper-sphere of unit radius in the space of quaternions.

Now, given that Eulero's parameters can be represented via quaternions, we can use quaternion algebra in order to handle rotations of reference frames.
In detail, let assume the following hypothesis:

- O1 and W are two reference frames with arbitrary rotation,
- $\mathbf{q}_{o1,w}$ is the quaternion which describes the rotation of O1 respect to W, and $\mathbf{q'}_{o1,w}$ is its conjugate.
- $\mathbf{p}_{p\text{-}o1,o1}$ is a pure quaternion where the vectorial-imaginary part is built with the position vector of the point P respect to the reference O1, in the coordinate system of O1, that is $\mathbf{p}_{p\text{-}o1,o1} = [0, \mathbf{P}_{o1}]$
- $\mathbf{p}_{p\text{-}w,w}$ is a pure quaternion where the vectorial-imaginary part is built with the position vector of the point P respect to the reference W, in the coordinate system of W, that is $\mathbf{p}_{p\text{-}w,w} = [0, \mathbf{P}_{w}]$

One could demonstrate the following property [5,9] :

$$\mathbf{p}_{p-w,w} = \mathbf{q}_{o1,w} \cdot \mathbf{p}_{p-o1,o1} \cdot \mathbf{q}'_{o1,w} \tag{13}$$

where quaternion product (eq.10) is used to obtain the same result of the usual alignment transformation with linear algebra:

$$\mathbf{P}_{p-w,w} = [\Lambda_{o1,w}]\mathbf{P}_{p-o1,o1} \tag{14}$$

where $\mathbf{P}$ is the three-dimensional vector of point position and $[\Lambda]$ is the 3x3 rotation matrix.

It is interesting to observe that, whenever a quaternion $\mathbf{q}$ represents a rotation, its conjugate $\mathbf{q'}$ represent the rotation in the opposite direction, therefore the inverse alignment-transformation

$$\mathbf{P}_{p-o1,o1} = [\Lambda_{o1,w}]^T \mathbf{P}_{p-w,w} \tag{15}$$

is simply obtained by conjugating the $\mathbf{q}$ quaternions of eq.13, that is:

$$\mathbf{p}_{p-o1,o1} = \mathbf{q}'_{o1,w} \cdot \mathbf{p}_{p-w,w} \cdot \mathbf{q}_{o1,w} \tag{16}$$

An useful side note is the following: if $\mathbf{q}$ is a rotation quaternion, the result of the multiplication by its conjugate is $\mathbf{qq'} = \{1,0,0,0\}$, which represent no rotation at all, in agreement with the fact that two rotations on the same axis but with opposite direction lead to no rotation at all.

Another interesting property is the concatenation of quaternion products to express concatenations of rotations (that is, coordinate transformation of points in a chain of reference frames). Say $[\Lambda_{o1,w}]$, $[\Lambda_{o2,o1}]$ and $[\Lambda_{o1,w}]$ are the relative rotation matrices of three references 01, 02, 03 in a chain of cartesian references W-O1-O2-O3, then:

$$\mathbf{P}_{p-w,w} = [\Lambda_{o1,w}][\Lambda_{o2,o1}][\Lambda_{o3,o2}]\mathbf{P}_{p-o3,o3} \tag{17}$$

can be expressed with quaternion algebra by way of the following multiplication:

$$\mathbf{p}_{p-w,w} = \mathbf{q}_{o1,w}\left(\mathbf{q}_{o2,o1}\left(\mathbf{q}_{o3,o2}\mathbf{p}_{p-o1,o1}\cdot\mathbf{q}'_{o3,2}\right)\mathbf{q}'_{o2,1}\right)\mathbf{q}'_{o1,w} \tag{18}$$

that is, taking advantage of the associative property of quaternion multiplication:

$$\mathbf{p}_{p-w,w} = \left(\mathbf{q}_{o1,w}\mathbf{q}_{o2,o1}\mathbf{q}_{o3,o2}\right)\mathbf{p}_{p-o1,o1}\left(\mathbf{q}'_{o3,2}\,\mathbf{q}'_{o2,1}\,\mathbf{q}'_{o1,w}\right) \tag{19}$$

hence, in general for a chain of *n* cartesian references,

$$\mathbf{p}_{p-w,w} = \mathbf{q}_{chain}\cdot\mathbf{p}_{p-o1,o1}\cdot\mathbf{q}'_{chain} \quad \text{with} \quad \mathbf{q}_{chain} = \left(\mathbf{q}_{o1,w}\cdot\ldots\cdot\mathbf{q}_{i,i-1}\cdot\ldots\cdot\mathbf{q}_{n,n-1}\right) \tag{20}$$

Note that the product of quaternions with unitary norm results in a quaternion with unitary norm, thus still belonging to the subset of Eulero’s parameters.

Also, it is common knowledge that the order of rotation transformations is non-commutative, just like the quaternion multiplication is a non-commutative operation (see equation 20).

Among other interesting properties of quaternion algebra applied to mechanics, there is the following relation, which obtains the time derivative of a rotation quaternion once the angular speed vector is known [5]:

$$\dot{\mathbf{q}}_{o1,w} = \frac{1}{2}\mathbf{w}_{o1,w}\cdot\mathbf{q}_{o1,w} \quad \text{with} \quad \mathbf{w}_{o1,w} = \left[0, \boldsymbol{\omega}_{o1,w}\right] \tag{21}$$

where the *pure quaternion* $\mathbf{w}_{o1,w}$ is built with the three-dimensional vector $\boldsymbol{\omega}_{o1,w}$, the angular speed of reference O1 respect to reference W, expressed in the coordinate system of W.

Also, the second time derivative can be obtained as well [13], if the angular acceleration vector $\boldsymbol{\alpha}$ is known, as expressed in the coordinate system W:

$$\ddot{\mathbf{q}}_{o1,w} = \frac{1}{2}\mathbf{a}_{o1,w}\cdot\mathbf{q}_{o1,w} \quad \text{with} \quad \mathbf{a}_{o1,w} = \left[0, \boldsymbol{\alpha}_{o1,w}\right] \tag{22}$$

4 CONSTRAINTS IN DYNAMICS AND KINEMATICS

In a multibody system based on cartesian coordinates, all constraints equations are coupled to the differential dynamical equations, resulting into a DAE system.
Constraints can be represented with a vector of equations of the type:

$$\mathbf{C}(\mathbf{q}, t) = \mathbf{0} \tag{23}$$

The dependence from coordinates $\mathbf{q}$ means that the constraints are *holonomic* (also known as “geometric” constraints), and the dependence from time –if any- is accountable of the definition *rheonomic*[1,3].

An easy and common way to solve this kind of system is to reduce it to an ODE (a set of ordinary differential equations). This implies that equation 23 must be differentiated twice with respect to time[1],

$$[C_q]\dot{\mathbf{q}} + \mathbf{C}_t = \mathbf{0} \tag{24}$$

$$[C_q]\ddot{\mathbf{q}} + 2 \cdot [C_{qt}]\dot{\mathbf{q}} + [C_{qq}]\dot{\mathbf{q}} + \mathbf{C}_{tt} = \mathbf{0} \tag{25}$$

shortly $[C_q]\ddot{\mathbf{q}} = \mathbf{Q}_c$ with $\mathbf{Q}_c = -2 \cdot [C_{qt}]\dot{\mathbf{q}} - [C_{qq}]\dot{\mathbf{q}} - \mathbf{C}_{tt}$ (26)

hence the ODE system:

$$\begin{bmatrix} [M] & [C_q]^T \\ [C_q] & [0] \end{bmatrix} \cdot \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{Q}} + \mathbf{Q}_m \\ \mathbf{Q}_c + \mathbf{Q}_s \end{Bmatrix} \tag{27}$$

where [M] is the mass matrix (mostly diagonal), $\mathbf{q}$ are the generalized coordinates, $\hat{\mathbf{Q}}$ is the vector of generalized lagrangian forces, $\mathbf{Q}_m$ is the vector of known inertial terms, $\mathbf{Q}_s$ is the vector of Baumgarte stabilizers which keep solutions on the $\mathbf{C}(\mathbf{q},t)=\mathbf{0}$ manifold as in complete DAE methods[6], $\boldsymbol{\lambda}$ is the vector of Lagrangian multipliers, $[C_q]$ is the jacobian of the constraint equations, and $\mathbf{Q}_c$ are the known terms of constraint equations, as in eq.26.

The calculus of the terms $[C_q]$, C_t and $\mathbf{Q}_c$ can heavily affect the speed of the simulation, so it may prove useful to find a straightforward analytical formulation instead of merely getting them with numerical differentiation.

5 CONSTRAINT EQUATIONS

Let consider the generic circumstance of a constraint where all the six mutual degrees of freedom of two rigid bodies are constrained with motion laws. These motion laws describe the reciprocal motion and rotation of the two bodies, so we must add two auxiliary reference frames on them, as in figure 1, and we call them “markers”.

As shown in picture, the two bodies are labeled O1 and O2, while the respective markers are labeled P and S.

For sake of generality, we suppose also that markers may have their own motion laws with respect to the parent bodies –if no laws are provided, the markers move firmly with rigid bodies-.

To set up this kind of link, thereafter nicknamed as “*lock constraint*”, we must write the equations that constraint the motion of marker P (belonging to body O1) respect to the marker S (belonging to body O2), in the coordinate system of marker S.

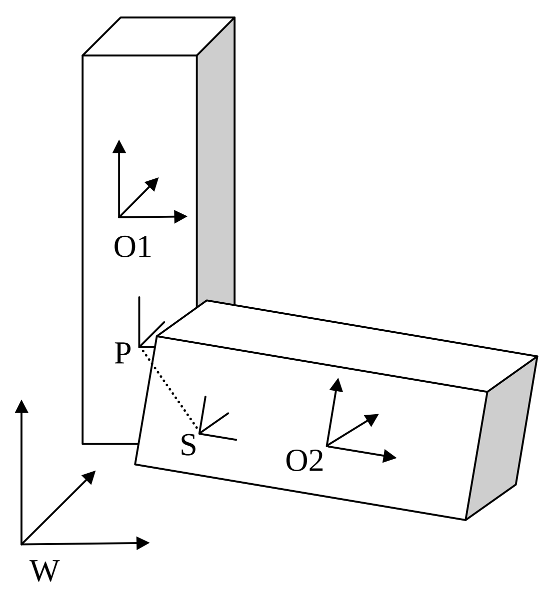


Figure 1: rigid bodies and markers

These constraint equations can be split in the translational and rotational parts, and must take into account the motion laws of relative translation/rotation of P respect to S, expressed in coordinate system of S, if any motion is needed.

4.1 Translational constraint

This constraint is considered in a three-dimensional vectorial form, and expresses the condition that the origin of P marker must follow a given trajectory respect to the S marker, in the coordinate system of S.

$$\mathbf{C} = \mathbf{q}_{P-S,S} - \mathbf{q}_{\Delta} = \mathbf{0} \tag{28}$$

being $\mathbf{q}_{P-S,S}$ the vector of markers' relative position, and $\mathbf{q}_{\Delta}$ the imposed motion law, in xyz space of S, that is $\mathbf{q}_{\Delta} = \mathbf{q}_{\Delta}(t)$. Note: if $\mathbf{q}_{\Delta} = \mathbf{0}$, the origins of P and S must superimpose.
By substituting the formulation of relative position P-S, one gets:

$$\mathbf{C} = \left[\Lambda_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \left(\left(\mathbf{q}^x_{O1,W} + \left[\Lambda_{O1}\right] \cdot \mathbf{u}_P\right) - \left(\mathbf{q}^x_{O2,W} + \left[\Lambda_{O2}\right] \cdot \mathbf{u}_S\right)\right) - \mathbf{q}_{\Delta} \tag{29}$$

where $\mathbf{u}_P$ and $\mathbf{u}_S$ are the positions of markers P and S about the coordinate systems of their bodies O1 and O2, respectively, and may be function of time themselves (generally these are constant).
Performing a differentiation with respect to time:

$$\dot{\mathbf{C}} = \dot{\mathbf{q}}_{PS-S} - \dot{\mathbf{q}}_{\Delta} = \mathbf{0} \tag{30}$$

where $\dot{\mathbf{q}}_{\Delta} = \dot{\mathbf{q}}_{\Delta}(t)$ is the time derivative of the motion law.
Keeping in mind that the term $\dot{\mathbf{q}}_{P-S,S}$ is the relative speed of P about S, in S coordinates, after some passages we get:

$$\dot{\mathbf{C}} = \left[\dot{\Lambda}_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \mathbf{q}_{P-S,W} + \left[\Lambda_{S,O2}\right]^T \cdot \left[\dot{\Lambda}_{O2}\right]^T \cdot \mathbf{q}_{P-S,W} + \left[\Lambda_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \dot{\mathbf{q}}_{P-S,W} - \dot{\mathbf{q}}_{\Delta}$$

$$\begin{aligned} \dot{\mathbf{C}} = {} & \left[\dot{\Lambda}_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \left(\left(\mathbf{q}^x_{O1} + \left[\Lambda_{O1}\right]\mathbf{u}_P\right) - \left(\mathbf{q}^x_{O2} + \left[\Lambda_{O2}\right]\mathbf{u}_S\right)\right) + \\ & + \left[\Lambda_{S,O2}\right]^T \cdot \left[\dot{\Lambda}_{O2}\right]^T \cdot \left(\left(\mathbf{q}^x_{O1} + \left[\Lambda_{O1}\right]\mathbf{u}_P\right) - \left(\mathbf{q}^x_{O2} + \left[\Lambda_{O2}\right]\mathbf{u}_S\right)\right) + \\ & + \left[\Lambda_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \left(\left(\dot{\mathbf{q}}^x_{O1} + \left[\dot{\Lambda}_{O1}\right]\mathbf{u}_P + \left[\Lambda_{O1}\right]\dot{\mathbf{u}}_P\right) - \left(\dot{\mathbf{q}}^x_{O2} + \left[\dot{\Lambda}_{O2}\right]\mathbf{u}_S + \left[\Lambda_{O2}\right]\dot{\mathbf{u}}_S\right)\right) - \dot{\mathbf{q}}_{\Delta} \end{aligned} \tag{31}$$

From the previous equation we can get also the $\mathbf{C}_t$ term which may be needed in inverse-kinematics procedures:

$$\mathbf{C}_t = \left[\dot{\Lambda}_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \mathbf{q}^x_{P-S,W} + \left[\Lambda_{S,O2}\right]^T \cdot \left(\left[\Lambda_{O2}\right]^T\left[\Lambda_{O1}\right]\dot{\mathbf{u}}_P - \dot{\mathbf{u}}_S\right) - \dot{\mathbf{q}}_{\Delta} \tag{32}$$

Performing a further differentiation with respect to time, we get:

$$\ddot{\mathbf{C}} = \ddot{\mathbf{q}}_{PS-S} - \ddot{\mathbf{q}}_{\Delta} = \mathbf{0} \tag{33}$$

where $\dot{\mathbf{q}}_{\Delta} = \dot{\mathbf{q}}_{\Delta}(t)$ is the acceleration of the motion law (known, and imposed by the user), while $\ddot{\mathbf{q}}_{P-S,S}$ is the relative acceleration of P about S. Knowing the formulation of such acceleration, we get:

$$\begin{aligned}\ddot{\mathbf{C}} = &\left[\ddot{\Lambda}_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \mathbf{q}_{P-S,W} + 2\left[\dot{\Lambda}_{S,O2}\right]^T \cdot \left[\dot{\Lambda}_{O2}\right]^T \cdot \mathbf{q}_{P-S,W} + 2\left[\dot{\Lambda}_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \dot{\mathbf{q}}_{P-S,W} + \\ &+ \left[\Lambda_{S,O2}\right]^T \cdot \left[\ddot{\Lambda}_{O2}\right]^T \cdot \mathbf{q}_{P-S,W} + 2\left[\Lambda_{S,O2}\right]^T \cdot \left[\dot{\Lambda}_{O2}\right]^T \cdot \dot{\mathbf{q}}_{P-S,W} + \left[\Lambda_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \ddot{\mathbf{q}}_{P-S,W} - \ddot{\mathbf{q}}_{\Delta}\end{aligned} \tag{34}$$

We must rework the previous equation in a form similar to eq.26, because the unknown terms in eq. 27 are the accelerations of bodies. With some algebraic manipulations, both the angular $\ddot{\mathbf{q}}_{\vartheta}$ and linear $\ddot{\mathbf{q}}_{x}$ accelerations can be put into evidence.

Introducing $\mathbf{Q}_{NA}$ for sake of compactness,

$$\begin{aligned}\mathbf{Q}_{NA} = &\left[\ddot{\Lambda}_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \mathbf{q}_{P-S,W} + 2\left[\dot{\Lambda}_{S,O2}\right]^T \cdot \left[\dot{\Lambda}_{O2}\right]^T \cdot \mathbf{q}_{P-S,W} + 2\left[\dot{\Lambda}_{S,O2}\right]^T \cdot \left[\Lambda_{O2}\right]^T \cdot \dot{\mathbf{q}}_{P-S,W} + \\ &+ 2\left[\Lambda_{S,O2}\right]^T \cdot \left[\dot{\Lambda}_{O2}\right]^T \cdot \dot{\mathbf{q}}_{P-S,W}\end{aligned} \tag{35}$$

substituting the formulation of relative acceleration $\ddot{\mathbf{q}}_{P-S,W}$,

$$\begin{aligned}\ddot{\mathbf{q}}_{P-S,W} = \ddot{\mathbf{q}}_{x_{P,W}} - \ddot{\mathbf{q}}_{x_{S,W}} = &\left(\ddot{\mathbf{q}}_{x_{O1,W}} + \left[\ddot{\Lambda}_{O1}\right] \cdot \mathbf{u}_P \cdot 2\left[\dot{\Lambda}_{O1}\right] \cdot \dot{\mathbf{u}}_P + \left[\Lambda_{O1}\right] \cdot \ddot{\mathbf{u}}_P\right) - \\ &- \left(\dot{\mathbf{q}}_{x_{O2,W}} + \left[\ddot{\Lambda}_{O2}\right] \cdot \mathbf{u}_S + 2\left[\dot{\Lambda}_{O2}\right] \cdot \dot{\mathbf{u}}_S + \left[\Lambda_{O2}\right] \cdot \ddot{\mathbf{u}}_S\right)\end{aligned} \tag{36}$$

and remembering the following relations,

$$\left[\ddot{\Lambda}\right] = [\Lambda][\hat{\alpha}] + [\Lambda][\hat{\omega}][\hat{\omega}] \quad , \quad [\hat{\alpha}] \cdot \mathbf{u} = -[\hat{u}] \cdot \boldsymbol{\alpha} \quad \text{and} \quad \boldsymbol{\alpha} = \left[Gl(\mathbf{q}_{\vartheta})\right] \cdot \ddot{\mathbf{q}}_{\vartheta} \tag{37}$$

we finally obtain:

$$\begin{aligned}\mathbf{C} = &\left[\Lambda_{S,O2}\right]^T \left[\Lambda_{O2}\right]^T \mathbf{q}_{x_{P-S,W}} + \\ &+ \left[\Lambda_{S,O2}\right]^T \left[\Lambda_{O2}\right]^T \begin{pmatrix} \mathbf{q}_{x_{O1,W}} - \left[\Lambda_{O1}\right]\left[\mathbf{u}_P\right]\left[Gl_{O1}\right]\mathbf{q}_{\vartheta_{O1,W}} + \\ + \left[\Lambda_{O1}\right]\left[\omega_{O1}\right]\left[\omega_{O1}\right] \cdot \mathbf{u}_P + 2\left[\Lambda_{O1}\right] \cdot \mathbf{u}_P + \left[\Lambda_{O1}\right] \cdot \mathbf{u}_P \end{pmatrix} - \\ &- \left[\Lambda_{S,O2}\right]^T \left[\Lambda_{O2}\right]^T \begin{pmatrix} \mathbf{q}_{x_{O2,W}} - \left[\Lambda_{O2}\right]\left[\mathbf{u}_S\right]\left[Gl_{O2}\right]\mathbf{q}_{\vartheta_{O2,W}} + \\ + \left[\Lambda_{O2}\right]\left[\omega_{O2}\right]\left[\omega_{O2}\right] \cdot \mathbf{u}_S + 2\left[\Lambda_{O2}\right] \cdot \mathbf{u}_S + \left[\Lambda_{O2}\right] \cdot \mathbf{u}_S \end{pmatrix} + \\ &+ \mathbf{Q}_{NA} - \mathbf{q}_{\Delta}\end{aligned} \tag{38}$$

We still must manipulate the term $\left[\Lambda_{S,O2}\right]^{T}\cdot\left[\Lambda_{O2}\right]^{T}\cdot\mathbf{q}_{P-S,W}$ in order to put into evidence the acceleration terms. In fact, remembering $\left[\ddot{\Lambda}\right]=[\Lambda][\hat{\alpha}]+[\Lambda][\hat{\omega}][\hat{\omega}]$ and the properties of hemisymmetric matrices, we get:

$$
\begin{aligned}
\mathbf{C} = & \left[\Lambda_{S,O2}\right]^{T}\left[\left[\Lambda_{O2}\right]\left[\omega_{O2}\right]\left[\omega_{O2}\right]\right]^{T}\mathbf{q}^{x}{}_{P-S,W} + \left[\Lambda_{S,O2}\right]^{T}\left[\widehat{\left\{\left[\Lambda_{O2}\right]^{T}\mathbf{q}_{P-S,W}\right\}}\right]\cdot\left[Gl_{O2}\right]\cdot\mathbf{q}^{\vartheta}{}_{O2,W} + \\
& + \left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T}\begin{pmatrix}\mathbf{q}^{x}{}_{O1,W} - \left[\Lambda_{O1}\right]\left[\mathbf{u}_{P}\right]\left[Gl_{O1}\right]\mathbf{q}^{\vartheta}{}_{O1,W} + \\ \left[\Lambda_{O1}\right]\left[\omega_{O1}\right]\left[\omega_{O1}\right]\cdot\mathbf{u}_{P} + 2\left[\Lambda_{O1}\right]\cdot\mathbf{u}_{P} + \left[\Lambda_{O1}\right]\cdot\mathbf{u}_{P}\end{pmatrix} + \\
& - \left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T}\begin{pmatrix}\mathbf{q}^{x}{}_{O2,W} - \left[\Lambda_{O2}\right]\left[\mathbf{u}_{S}\right]\left[Gl_{O2}\right]\mathbf{q}^{\vartheta}{}_{O2,W} + \\ \left[\Lambda_{O2}\right]\left[\omega_{O2}\right]\left[\omega_{O2}\right]\cdot\mathbf{u}_{S} + 2\left[\Lambda_{O2}\right]\cdot\mathbf{u}_{S} + \left[\Lambda_{O2}\right]\cdot\mathbf{u}_{S}\end{pmatrix} + \\
& + \left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T}\mathbf{q}^{x}{}_{P-S,W} + 2\left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T}\mathbf{q}^{x}{}_{P-S,W} + 2\left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T}\mathbf{q}^{x}{}_{P-S,W} \\
& + 2\left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T}\mathbf{q}^{x}{}_{P-S,W} - \mathbf{q}_{\Delta}
\end{aligned}
\tag{39}
$$

In the previous equation, the acceleration terms can be put in evidence, thus getting a formula in the form of eq.26, that is $\left[C_{q}\right]\ddot{\mathbf{q}}=\mathbf{Q}_{c}$.

Hence, introducing the vector which contains the angular and linear accelerations of both bodies $\ddot{\mathbf{q}}^{v}{}_{O1\&O2}=\left\{\ddot{\mathbf{q}}^{x}{}_{O1,W}\;\ddot{\mathbf{q}}^{\vartheta}{}_{O1,W}\;\ddot{\mathbf{q}}^{x}{}_{O2,W}\;\ddot{\mathbf{q}}^{\vartheta}{}_{O2,W}\right\}^{T}$, we can write:

$$\left[Cx_{q}\right]\ddot{\mathbf{q}}^{v}{}_{O1\&O2}=\mathbf{Q}_{c} \tag{40}$$

where the jacobian [Cx $_q$] is computed piecewise in the following way:

$$\left[Cx_{q}\right]=\left[\left[Cx_{q}\right]_{xO1}\quad\left[Cx_{q}\right]_{\vartheta O1}\quad\left[Cx_{q}\right]_{xO2}\quad\left[Cx_{q}\right]_{\vartheta O2}\right] \tag{41}$$

given that each part of that jacobian can be easily recovered from equation 39:

$$\left[Cx_{q}\right]_{xO1}=+\left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T} \tag{42 a}$$
$$\left[Cx_{q}\right]_{\vartheta O1}=-\left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T}\left[\Lambda_{O1}\right]\left[\hat{\mathbf{u}}_{P}\right]\left[Gl_{O1}\right] \tag{42 b}$$
$$\left[Cx_{q}\right]_{xO2}=-\left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T} \tag{42 c}$$
$$\left[Cx_{q}\right]_{\vartheta O2}=+\left[\Lambda_{S,O2}\right]^{T}\left[\Lambda_{O2}\right]^{T}\left[\Lambda_{O2}\right]\left[\hat{\mathbf{u}}_{S}\right]\left[Gl_{O2}\right]+\left[\Lambda_{S,O2}\right]^{T}\left[\widehat{\left\{\left[\Lambda_{O2}\right]^{T}\mathbf{q}_{P-S,W}\right\}}\right]\cdot\left[Gl_{O2}\right] \tag{42 d}$$

From eq.39 we get also the known term $\mathbf{Q}_{c}$:

$$\begin{aligned}
\mathbf{Q}_C =& \left[\Lambda_{S,O2}\right]^T\left[\left[\Lambda_{O2}\right]\left[\omega_{O2}\right]\left[\omega_{O2}\right]\right]^T \mathbf{q}_{x_{P-S,W}} + \\
&+\left[\Lambda_{S,O2}\right]^T\left[\Lambda_{O2}\right]^T\left(\left[\Lambda_{O1}\right]\left[\omega_{O1}\right]\left[\omega_{O1}\right]\cdot\mathbf{u}_P + 2\left[\Lambda_{O1}\right]\cdot\mathbf{u}_P + \left[\Lambda_{O1}\right]\cdot\mathbf{u}_P\right) + \\
&-\left[\Lambda_{S,O2}\right]^T\left[\Lambda_{O2}\right]^T\left(\left[\Lambda_{O2}\right]\left[\omega_{O2}\right]\left[\omega_{O2}\right]\cdot\mathbf{u}_S + 2\left[\Lambda_{O2}\right]\cdot\mathbf{u}_S + \left[\Lambda_{O2}\right]\cdot\mathbf{u}_S\right) + \\
&+\left[\Lambda_{S,O2}\right]^T\left[\Lambda_{O2}\right]^T\mathbf{q}_{x_{P-S,W}} + 2\left[\Lambda_{S,O2}\right]^T\left[\Lambda_{O2}\right]^T\mathbf{q}_{x_{P-S,W}} + 2\left[\Lambda_{S,O2}\right]^T\left[\Lambda_{O2}\right]^T\mathbf{q}_{x_{P-S,W}} + \\
&+2\left[\Lambda_{S,O2}\right]^T\left[\Lambda_{O2}\right]^T\mathbf{q}_{x_{P-S,W}} - \mathbf{q}_\Delta
\end{aligned} \tag{43}$$

Note: all the equations above require the knowledge ot the terms $\mathbf{q}_{x_{P-S,W}}, \dot{\mathbf{q}}_{x_{P-S,W}}$ (relative marker position and speed, in absolute reference W), which can be computed as follows:

$$\mathbf{q}_{P-S,W} = \mathbf{q}_{x_{P,W}} - \mathbf{q}_{x_{S,W}} = \left(\mathbf{q}_{x_{O1,W}} + \left[\Lambda_{O1}\right]\cdot\mathbf{u}_P\right) - \left(\mathbf{q}_{x_{O2,W}} + \left[\Lambda_{O2}\right]\cdot\mathbf{u}_S\right) \tag{44}$$

$$\dot{\mathbf{q}}_{P-S,W} = \dot{\mathbf{q}}_{x_{P,W}} - \dot{\mathbf{q}}_{x_{S,W}} = \left(\dot{\mathbf{q}}_{x_{O1,W}} + \left[\dot{\Lambda}_{O1}\right]\cdot\mathbf{u}_P + \left[\Lambda_{O1}\right]\cdot\dot{\mathbf{u}}_P\right) - \left(\dot{\mathbf{q}}_{x_{O2,W}} + \left[\dot{\Lambda}_{O2}\right]\cdot\mathbf{u}_S + \left[\Lambda_{O2}\right]\cdot\dot{\mathbf{u}}_S\right) \tag{45}$$

4.2 Rotational constraint

This constraint introduces the condition that the P marker must rotate about the S marker, with the motion law of rotation $\mathbf{q}_{\vartheta_\Delta} = \mathbf{q}_{\vartheta_\Delta}(t)$ expressed in the coordinate system of S.
This constraint is equivalent to the equation:

$$\left[\Lambda\left(\mathbf{q}_{\vartheta_\Delta}\right)\right]^T \cdot \left[\Lambda\left(\mathbf{q}_{\vartheta_{P-S,S}}\right)\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

Given that rotations can be expressed with quaternion multiplications, as described in eq.13 and eq 14, we can translate the previous constraint in quaternion algebra:

$$\mathbf{C} = \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_{P-S,S}} - \mathbf{q}_{\vartheta_{\Re e}} = \mathbf{0} \tag{47}$$

where the real quaternion $\mathbf{q}_{\vartheta_{\Re e}} = \{1,0,0,0\}^T$ expresses a null rotation just like the unitary diagonal 3x3 matrix of equation 46.
The term $\mathbf{q}_{\vartheta_\Delta}^{-1}$ is the inverse of the quaternion $\mathbf{q}_{\vartheta_\Delta}$ which comes from an imposed law of rotation $\mathbf{q}_{\vartheta_\Delta} = \mathbf{q}_{\vartheta_\Delta}(t)$. Note: since it is an unitary quaternion, the inverse is the same as the conjugate, $\mathbf{q}_{\vartheta_\Delta}^{-1} = \mathbf{q}'_{\vartheta_\Delta}$, which is easy to compute (eq.7).
The term $\mathbf{q}_{\vartheta_{P-S,S}}$ means the relative rotation of P about S, in coordinate system of S, expressed in quaternion algebra. We can find that its expression is

$$\mathbf{q}_{\vartheta_{P-S,S}} = \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} \tag{48}$$

as long as $[\Lambda_{P,S}] = [\Lambda_{S,O2}]^T[\Lambda_{O2,W}]^T[\Lambda_{O1,W}][\Lambda_{P,O1}]$. The quaternions $\mathbf{q}_{\vartheta_S}$ and $\mathbf{q}_{\vartheta_S}$ are the rotations of the two markers P and S about the frames of their bodies O1 and O2, and the quaternions $\mathbf{q}_{\vartheta_{O1}}$ and $\mathbf{q}_{\vartheta_{O2}}$ are the rotations of rigid bodies about the absolute frame W. Introducing the above equation into the formulation of the constraint, we get:

$$\mathbf{C} = \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} - \mathbf{q}_{\vartheta_{\Re e}} = \mathbf{0} \tag{49}$$

Applying symbolic differentiation with respect to time, and knowing that $\dot{\mathbf{q}}_{\vartheta_{\Re e}} = \mathbf{0}$, it turns into

$$\begin{aligned}\dot{\mathbf{C}} = {} & \dot{\mathbf{q}}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \\ & + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_P}\end{aligned} \tag{50}$$

From this result, we can extract also the term $\mathbf{C}_t$ which is often used for inverse kinematics:

$$\begin{aligned}\mathbf{C}_t = {} & \dot{\mathbf{q}}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \\ & + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_P}\end{aligned} \tag{51}$$

Performing a further differentiation with respect to time, we get:

$$\begin{aligned}\ddot{\mathbf{C}} = {} & \ddot{\mathbf{q}}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + 2\dot{\mathbf{q}}_{\vartheta_\Delta}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \\ & + 2\dot{\mathbf{q}}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + 2\dot{\mathbf{q}}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \\ & + 2\dot{\mathbf{q}}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_P} + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \ddot{\mathbf{q}}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \\ & + 2\mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_S}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + 2\mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \\ & + 2\mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_P} + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \ddot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + \\ & + 2\mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + 2\mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_P} + \\ & + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \ddot{\mathbf{q}}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_P} + 2\mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_P} + \\ & + \mathbf{q}_{\vartheta_\Delta}^{-1} \cdot \mathbf{q}_{\vartheta_S}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \ddot{\mathbf{q}}_{\vartheta_P}\end{aligned} \tag{52}$$

The previous expression involves 60 quaternion products. However it must be pointed out that, in most situations, such equation can be computed really fast: many of its addenda can be simplified if the markers P and S do not have their own motion laws about O1 and O2 (that is, if they are just fixed to the respective bodies, terms like $\dot{\mathbf{q}}_{\vartheta_S}, \ddot{\mathbf{q}}_{\vartheta_S}, \dot{\mathbf{q}}_{\vartheta_P}$ and $\ddot{\mathbf{q}}_{\vartheta_P}$ are null quaternions, thus leading to a much easier formulation of $\mathbf{C}$).

Further simplifications can be performed when no time-dependant rotations are imposed between P and S, hence $\dot{\mathbf{q}}_{\vartheta_\Delta}$ and $\ddot{\mathbf{q}}_{\vartheta_\Delta}$ are null as well, and just three addenda remain in eq.52.

Now, in order to put into evidence the body-acceleration terms, we must manipulate the equation with some algebra, as we already did for the linear constraint.
First of all, we group all the known terms into $\mathbf{Q}_c$:

$$\begin{aligned}
\mathbf{Q}c = &-\ddot{\mathbf{q}}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} - 2\dot{\mathbf{q}}_{\vartheta_\Delta}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} + \\
&- 2\dot{\mathbf{q}}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} - 2\dot{\mathbf{q}}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} + \\
&- 2\dot{\mathbf{q}}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\dot{\mathbf{q}}_{\vartheta_P} - \mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\ddot{\mathbf{q}}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} + \\
&- 2\mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_S}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} - 2\mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} + \\
&- 2\mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\dot{\mathbf{q}}_{\vartheta_P} - 2\mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_{O2}}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} + \\
&- 2\mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\dot{\mathbf{q}}_{\vartheta_P} - 2\mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\dot{\mathbf{q}}_{\vartheta_{O1}}\cdot\dot{\mathbf{q}}_{\vartheta_P} + \\
&- \mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\ddot{\mathbf{q}}_{\vartheta_P}
\end{aligned} \tag{53}$$

Then we have:

$$\mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\ddot{\mathbf{q}}_{\vartheta_{O2}}^{-1}\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} + \mathbf{q}_{\vartheta_\Delta}^{-1}\cdot\mathbf{q}_{\vartheta_S}^{-1}\cdot\mathbf{q}_{\vartheta_{O2}}^{-1}\cdot\ddot{\mathbf{q}}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} = \mathbf{Q}c \tag{54}$$

Quaternion products can be written also in form of linear algebra, that is:

$$\mathbf{q}_1\ \mathbf{q}_2 = (s_1 s_2 - \mathbf{v}_1\mathbf{v}_2,\quad s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

$$\mathbf{q}_1\ \mathbf{q}_2 = \left[\overset{+}{\mathbf{q}}_1\right]\cdot\mathbf{q}_2 \tag{55}$$

where we used the 4x4 matrix[14]:

$$\left[\overset{+}{\mathbf{q}}\right] = \begin{bmatrix} +q_0 & -q_1 & -q_2 & -q_3 \\ +q_1 & +q_0 & -q_3 & +q_2 \\ +q_2 & +q_3 & +q_0 & -q_1 \\ +q_3 & -q_2 & +q_1 & +q_0 \end{bmatrix} \tag{56}$$

Since quaternion product is non-commutative, $\mathbf{q}_1\ \mathbf{q}_2 \neq \left[\overset{+}{\mathbf{q}}_2\right]\cdot\mathbf{q}_1$, anyway we can write:

$$\mathbf{q}_1\ \mathbf{q}_2 = \left[\overset{+}{\mathbf{q}}_1\right]\cdot\mathbf{q}_2 = \left[\overset{-}{\mathbf{q}}_2\right]\cdot\mathbf{q}_1 \tag{57}$$

where we introduced another 4x4 matrix[14],

$$\left[\overset{-}{\mathbf{q}}\right] = \begin{bmatrix} +q_0 & -q_1 & -q_2 & -q_3 \\ +q_1 & +q_0 & +q_3 & -q_2 \\ +q_2 & -q_3 & +q_0 & +q_1 \\ +q_3 & +q_2 & -q_1 & +q_0 \end{bmatrix} \tag{58}$$

Thanks to the previous formula, we can transform equation 54 into:

$$\left[\overset{+}{\mathbf{q}}_{\vartheta_\Delta}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta_S}^{-1}\right]\cdot\left[\overset{+}{\ddot{\mathbf{q}}}_{\vartheta_{O2}}^{-1}\right]\cdot\mathbf{q}_{\vartheta_{O1}}\cdot\mathbf{q}_{\vartheta_P} + \left[\overset{+}{\mathbf{q}}_{\vartheta_\Delta}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta_S}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta_{O2}}^{-1}\right]\cdot\left[\overset{+}{\ddot{\mathbf{q}}}_{\vartheta_{O1}}\right]\cdot\mathbf{q}_{\vartheta_P} = \mathbf{Q}c \tag{59}$$

Using the property of eq.57, we can move the acceleration terms to the right of each addenda:

$$\left[\overset{+}{\mathbf{q}}_{\vartheta\Delta}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta S}^{-1}\right]\cdot\left[\overset{-}{\left\{\mathbf{q}_{\vartheta O1}\cdot\mathbf{q}_{\vartheta P}\right\}}\right]\cdot\ddot{\mathbf{q}}_{\vartheta O2}^{-1}+\left[\overset{+}{\mathbf{q}}_{\vartheta\Delta}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta S}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta O2}^{-1}\right]\cdot\left[\overset{-}{\mathbf{q}}_{\vartheta P}\right]\cdot\ddot{\mathbf{q}}_{\vartheta O1}=\mathbf{Q}c \tag{60}$$

However, in the first addendum we do not see O2 body's acceleration, but rather its conjugate. This problem is solved in a straightforward way, as soon as the conjugate of a quaternion will be expressed by means of linear algebra:

$$\mathbf{q}=[\chi_{\pm\Im}]\cdot\mathbf{q}^{-1} \tag{61}$$

where we introduce a new matrix

$$[\chi_{\pm\Im}]=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix} \tag{62}$$

Now we can readily write:

$$\left[\overset{+}{\mathbf{q}}_{\vartheta\Delta}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta S}^{-1}\right]\cdot\left[\overset{-}{\left\{\mathbf{q}_{\vartheta O1}\cdot\mathbf{q}_{\vartheta P}\right\}}\right]\cdot\left[\chi_{\pm\Im}\right]\cdot\ddot{\mathbf{q}}_{\vartheta O2}+\left[\overset{+}{\mathbf{q}}_{\vartheta\Delta}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta S}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta O2}^{-1}\right]\cdot\left[\overset{-}{\mathbf{q}}_{\vartheta P}\right]\cdot\ddot{\mathbf{q}}_{\vartheta O1}=\mathbf{Q}c \tag{63}$$

Introducing the vector which contains the angular and linear accelerations of both bodies $\ddot{\mathbf{q}}_{v\,O1\&O2}=\left\{\ddot{\mathbf{q}}_{x\,O1,w}\,\ddot{\mathbf{q}}_{\vartheta\,O1,w}\,\ddot{\mathbf{q}}_{x\,O2,w}\,\ddot{\mathbf{q}}_{\vartheta\,O2,w}\right\}^{T}$, we can write:

$$\left\lfloor C_{\vartheta_q}\right\rfloor\ddot{\mathbf{q}}_{v\,O1\&O2}=\mathbf{Q}_c \tag{64}$$

where the jacobian $[C_q]$ is computed piecewise in the following way:

$$\left\lfloor C_{\vartheta_q}\right\rfloor=\left\lfloor\left\lfloor C_{\vartheta_q}\right\rfloor_{xO1}\quad\left\lfloor C_{\vartheta_q}\right\rfloor_{\vartheta O1}\quad\left\lfloor C_{\vartheta_q}\right\rfloor_{xO2}\quad\left\lfloor C_{\vartheta_q}\right\rfloor_{\vartheta O2}\right\rfloor \tag{65}$$

and each part of that jacobian can be easily obtained from equation 63 (note that two matrices are null, since no linear acceleration terms appear in eq.63):

$$\left\lfloor C_{\vartheta_q}\right\rfloor_{xO1}=[0] \tag{66}$$

$$\left\lfloor C_{\vartheta_q}\right\rfloor_{\vartheta O1}=\left[\overset{+}{\mathbf{q}}_{\vartheta\Delta}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta S}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta O2}^{-1}\right]\cdot\left[\overset{-}{\mathbf{q}}_{\vartheta P}\right] \tag{67}$$

$$\left\lfloor C_{\vartheta_q}\right\rfloor_{xO2}=[0] \tag{68}$$

$$\left\lfloor C_{\vartheta_q}\right\rfloor_{\vartheta O2}=\left[\overset{+}{\mathbf{q}}_{\vartheta\Delta}^{-1}\right]\cdot\left[\overset{+}{\mathbf{q}}_{\vartheta S}^{-1}\right]\cdot\left[\overset{-}{\left\{\mathbf{q}_{\vartheta O1}\cdot\mathbf{q}_{\vartheta P}\right\}}\right]\cdot\left[\chi_{\pm\Im}\right] \tag{69}$$

Furthermore, the term **Q**c is:

$$
\begin{aligned}
\mathbf{Q}c = & -\ddot{\mathbf{q}}_{\vartheta_{\Delta}}^{-1} \cdot \mathbf{q}_{\vartheta_{S}}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_{P}} - 2\dot{\mathbf{q}}_{\vartheta_{\Delta}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{S}}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_{P}} + \\
& - 2\dot{\mathbf{q}}_{\vartheta_{\Delta}}^{-1} \cdot \mathbf{q}_{\vartheta_{S}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_{P}} - 2\dot{\mathbf{q}}_{\vartheta_{\Delta}}^{-1} \cdot \mathbf{q}_{\vartheta_{S}}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_{P}} + \\
& - 2\dot{\mathbf{q}}_{\vartheta_{\Delta}}^{-1} \cdot \mathbf{q}_{\vartheta_{S}}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_{P}} - \mathbf{q}_{\vartheta_{\Delta}}^{-1} \cdot \ddot{\mathbf{q}}_{\vartheta_{S}}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_{P}} + \\
& - 2\mathbf{q}_{\vartheta_{\Delta}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{S}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_{P}} - 2\mathbf{q}_{\vartheta_{\Delta}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{S}}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_{P}} + \\
& - 2\mathbf{q}_{\vartheta_{\Delta}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{S}}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_{P}} - 2\mathbf{q}_{\vartheta_{\Delta}}^{-1} \cdot \mathbf{q}_{\vartheta_{S}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O1}} \cdot \mathbf{q}_{\vartheta_{P}} + \\
& - 2\mathbf{q}_{\vartheta_{\Delta}}^{-1} \cdot \mathbf{q}_{\vartheta_{S}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_{P}} - 2\mathbf{q}_{\vartheta_{\Delta}}^{-1} \cdot \mathbf{q}_{\vartheta_{S}}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \dot{\mathbf{q}}_{\vartheta_{O1}} \cdot \dot{\mathbf{q}}_{\vartheta_{P}} + \\
& - \mathbf{q}_{\vartheta_{\Delta}}^{-1} \cdot \mathbf{q}_{\vartheta_{S}}^{-1} \cdot \mathbf{q}_{\vartheta_{O2}}^{-1} \cdot \mathbf{q}_{\vartheta_{O1}} \cdot \ddot{\mathbf{q}}_{\vartheta_{P}}
\end{aligned}
\qquad (70)
$$

4.3 Complete "lock" constraint

If we consider the complete "lock" constraint, which includes both the constraint on mutual translation and mutual rotation of the markers P and S, we get a jacobian matrix which is built by stacking the two jacobians of eq.41 and eq.65, one over the other:

$$
\left[C_q\right]_{lock} = \begin{bmatrix} \left[C_{x_q}\right]_{xO1} & \left[C_{x_q}\right]_{\vartheta O1} & \left[C_{x_q}\right]_{xO2} & \left[C_{x_q}\right]_{\vartheta O2} \\ \left[C_{\vartheta_q}\right]_{xO1} & \left[C_{\vartheta_q}\right]_{\vartheta O1} & \left[C_{\vartheta_q}\right]_{xO2} & \left[C_{\vartheta_q}\right]_{\vartheta O2} \end{bmatrix} \qquad (71)
$$

The submatrices of the first row (referring to the constraint on translations) are expressed by eq.42 and following, while the sub-matrices of the second row (pertaining to the rotational constraint) are expressed by eq.66 and following.

This jacobian has 7 rows (descending from 3 conditions on mutual movement plus 4 conditions of the quaternion equation for rotation) and 14 columns (because both body frames have 7 coordinates: 3 cartesian and 4 for the quaternion representing the rotation).

However, classical mechanics teaches that six equations are enough to restraint the relative degrees of freedom of two rigid bodies, while our jacobian has seven rows: one more than the strictly needed.

This happens because one of the four constraints descending from the quaternion equation is redundant. In fact, if we had already introduced in the DAE system N constraints about the unit length of the rotation quaternions of the N rigid bodies, the product of eq.48 should by sure return a quaternion of unit length: given 3 free components, the fourth would follow immediately because of this unit-length restraint.

With this assumption, one of the four conditions expressed by the quaternion constraint is redundant and can be eliminated [13], thus getting a jacobian with the six rows (3+3) which are

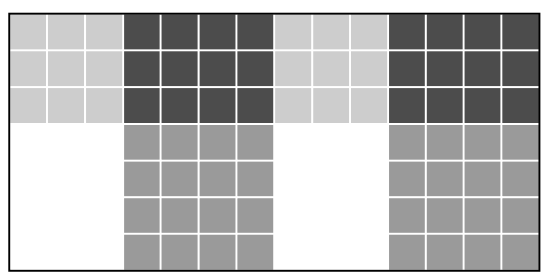


Figure 2: aspect of the 7x14 jacobian matrix of the complete "lock" constraint

strictly needed.

Hence the resulting “lock” formulation spawn a vector of six restraint equations, the first three coming from the cartesian constraint of eq28, and the other three coming from three components of the quaternion-based rotational constraint of eq.49 (an opportune choice is to select the vectorial-imaginary part):

$$\mathbf{C}_{\mathrm{lock}} = \{\mathbf{C}_X \quad \Im\{\mathbf{C}_\theta\}\}^T \quad , \quad \mathbf{C}_{\mathrm{lock}} = \{C_x \quad C_y \quad C_z \quad C_{\theta i} \quad C_{\theta j} \quad C_{\theta k}\}^T \qquad (72)$$

In a similar fashion, we get also $\dot{\mathbf{C}}_{\mathrm{lock}}$, $\ddot{\mathbf{C}}_{\mathrm{lock}}$, $\mathbf{Q}_{c\,\mathrm{lock}}$, $\mathbf{C}_{t\,\mathrm{lock}}$.

6 OTHER CONSTRAINTS

Heretofore we introduced the *lock* formalism which represents constraints where all the 6 relative degrees of freedom of two bodies are restrained, occasionally with motion laws.

If we suppress one or more of the 6 constraint equations, some reciprocal movements are left free and we can create different types of joints with enough physical interpretation (revolute joints, prismatic guides, spherical joints, etc.).

Also, if we provide adequate motion laws, we can use the same lock formalism to simulate engines, linear actuators, assignment of trajectories, and so on.

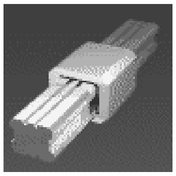
From a programmer’s point of view, this means that for all the links in the multibody system the “lock” formulation is computed to get the complete $[C_q]_{\mathrm{lock}}$, $\mathbf{Q}_{C\mathrm{lock}}$, $\mathbf{C}_{\mathrm{lock}}$, $\mathbf{C}_{t\,\mathrm{lock}}$ vectors and matrices, but only selected rows of the jacobian (and the corresponding elements in the $\mathbf{Q}_{C\,\mathrm{lock}}$, $\mathbf{C}_{\mathrm{lock}}$, $\mathbf{C}_{t\,\mathrm{lock}}$ vectors) are used and pasted into the DAE system.

In this paper, as practical examples, we take into account only the most meaningful joints among all the 64 possible variants which can be obtained by suppressing different equations of the lock formulation.

6.1 Spherical joint

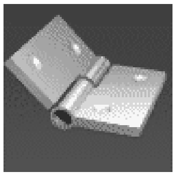
This joint can be obtained, of course, with the simple suppression of all the 3 constraints about mutual rotation; only the cartesian C_x,C_y,C_z constraints are left. The resulting jacobian matrix has only three rows, extracted from row 1,2,3 of the $[C_q]_{\mathrm{lock}}$ matrix of eq.71. Also the vectors $\mathbf{Q}$c, $\mathbf{C}$, $\mathbf{C}$t have only three elements.

6.2 Prismatic joint



This joint means the suppression of only one of the three cartesian constraints, for example the elimination of the C_z constraint allows the shifting of the marker P along the Z axis of marker S (hence the Z axis of S would be used to indicate the direction of the prismatic joint). On the other hand, all the three rotational constraints are kept active. The resulting jacobian matrix has five rows, extracted from rows 1,2,4,5,6 of the $[C_q]_{\mathrm{lock}}$ matrix of eq.71

6.3 Revolute joint



The revolute joint implies the superposition of marker's origins, so all three cartesian constraints C_x, C_y, C_z are kept active. Assuming that this joint allows the rotation of marker P about axis Z of marker S, one of the three rotational constraints must be eliminated. It is easy to find that, if a rotation is performed about versor Z, the X and Y components of the vectorial part of the resulting quaternion are null. Hence, the two active rotational constraints are $C_{\theta i}$ and $C_{\theta j}$, while $C_{\theta k}$ is not taken into consideration.
The jacobian matrix has five rows extracted from row 1,2,3,4,5 of the $[C_q]_{lock}$ matrix.

6.4 Cylindrical joint

This joint is similar to the revolute joint, but allows also the shifting of marker P respect along the Z axis of marker S . Shifting in X and Y is forbidden, and only rotation about Z is allowed.
The jacobian matrix has four rows, extracted from row 1,2,4,5 of the $[C_q]_{lock}$ matrix.

6.5 Engines, motors

A spinning engine can be represented by all the 6 constraint equations of the "lock" formulation, where a user-defined motion law has been defined for the mutual rotation of the two markers P and S, thus computing all formulas with the specified $\mathbf{q}_{\vartheta_\Delta} = \mathbf{q}_{\vartheta_\Delta}(t)$ function and its derivatives.

Otherwise, $\mathbf{q}_{\vartheta_\Delta}$ could be kept constant and the motion law could be applied to the terms $\mathbf{q}_{\vartheta_P} = \mathbf{q}_{\vartheta_P}(t)$ or $\mathbf{q}_{\vartheta_S} = \mathbf{q}_{\vartheta_S}(t)$, which represent the rotations of markers respect to their rigid bodies.

6.5 Other examples

In table 1 we report some examples of joints which can be easily obtained from the "*lock*" formalism. The "X" symbol means 'active constraint equation'.
Note that constraining just one of the three rotational degrees results in a joint which transmits rotation in a homokinetic fashion, like the Birfield or Rzeppa devices.

	C_x	C_y	C_z	$C_{\theta i}$	$C_{\theta j}$	$C_{\theta k}$
Bolt / glue / fastener / nail / etc..	x	x	x	x	x	x
Point on line	x	x				
Point on plane			x			
Plane on plane			x	x	x	
Revolute	x	x	x	x	x	
Cylindrical joint	x	x		x	x	
Angular alignment				x	x	x
Oldham joint	x			x	x	x
Prismatic joint	x	x		x	x	x
Birfield or Rzeppa Joint	x	x				x
homokinetic joint	x	x	x			x

Table 1: some examples of joints inherited from the "lock" constraint

CONCLUSIONS

The adoption of quaternion as rotational coordinates allows a compact and versatile formulation of constraint equations. Taking advantage of quaternion algebra, we developed a formalism which exploits high generality, since it deals with the circumstance of constraints between markers where rheonomic laws can be assigned either to the mutual translation/rotation, either to the translation/rotation of markers respect to parent bodies. Henceforth, many special purpose joints can be obtained from that single vectorial formulation, just by suppression of constraint scalar equations, and by providing adequate motion laws when a rheonomic behavior is needed.

We accomplished the analytical derivations of the constraint equations in order to avoid numerical computation of jacobians, thus getting superior speed and precision.

Despite the apparent complexity of the analytical derivations, most formulas can be simplified on the basis of the features used in the joint (presence of motion laws, etc.) and run-time optimizations can take place during numerical calculus of equations.

These theoretical results fit well into an object-oriented approach to the programming of multibody software. We developed in this sense our multibody software CHRONO, which indeed shows high speed of calculus and stimulates further research in this field.

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